Journal of Statistical Physics, Vol. 15, No. 5, 1976

Studies in Nonlinear Stochastic Processes. II. The Duffing Oscillator Revisited

A. B. Budgor,[1] K. Lindenberg,[1] and K. E. Shuler[1]

Received April 27, 1976

We have applied the approximation method of statistical linearization and various higher order corrections thereto to the study of a nonlinear oscillator perturbed by Gaussian, delta-correlated noise. We compute the second-order statistics of the response, i.e., the variances, autocorrelation functions, and spectral densities for various forms of the nonlinearity and compare our results with the few more exact calculations which are available in the literature. We show that a very simple modification of statistical linearization, based upon the use of the variance as obtained from the appropriate Fokker–Planck equation, yields results which are in better agreement with the "exact" literature results than either statistical linearization or first-order corrections thereto. This modified method of statistical linearization has the significant advantage of great computational simplicity as compared to other attempts of accurate calculations of second-order statistics of nonlinear stochastic equations now in the literature.

KEY WORDS: Stochastic processes; nonlinear stochastic equations; statistical linearization; autocorrelation functions; spectral densities.

1. INTRODUCTION

In the first paper of this series,[1] Budgor presented a general exposition of the method of "statistical linearization" and its higher order correction terms for the approximate solution of nonlinear stochastic differential equations.

In this paper, we apply the method of statistical linearization to a study of the "stochastic" Duffing oscillator, i.e., an anharmonic damped oscillator

This work was supported in part by the National Science Foundation under Grants MPS 72-04363 and CHE 75-20624.

[1] Department of Chemistry, University of California—San Diego, La Jolla, California.

subject to random excitations. The Duffing oscillator plays the analogous role in the study of nonlinear systems as the harmonic oscillator does for linear ones—it is one of the simplest systems which can be treated in sufficient detail to investigate the usefulness of approximation methods. For this reason, the Duffing oscillator has already been discussed in detail by Caughey[2] and Crandall.[3] Our reasons, and excuse, for presenting here one more study of this system are fourfold: (1) to introduce this useful method to the chemical and statistical physics fraternity, (2) to compare the statistical linearization results including higher correction terms with recent results on the Duffing oscillator obtained by an entirely different method by Morton and Corrsin[4] (MC) and Bixon and Zwanzig,[5] (3) to present some new results on a simple but useful correction to statistical linearization for the Duffing oscillator, and (4) to extend some of these results to higher nonlinearities. Items 2–4 are not covered by the previously cited work.

In Section 2 we show how to apply the method of statistical linearization to the Duffing oscillator. In Section 3 we develop higher order corrections to statistical linearization to obtain explicit expressions for the variance, the autocorrelation function, and the spectral density for the position variable $x(t)$ as a function of the damping parameter and the form of the nonlinearity. In Section 4 we present via tables and graphs computer calculations of these functions based on the analytical expressions of Sections 2 and 3 and compare these results with the calculations of Morton and Corrsin. We also develop in this section a new, simple, and useful correction to statistical linearization which considerably improves the agreement with the Morton–Corrsin calculations. In Section 5 we briefly summarize and discuss our results.

2. STATISTICAL LINEARIZATION

The method of statistical linearization is based on the replacement of nonlinear stochastic equations which cannot be solved analytically by "equivalent" linear equations. These linear equations are chosen so as to minimize, in some average statistical sense, the error made by this replacement.

A general discussion of the method of statistical linearization and systematic higher order corrections thereto is given in a companion paper,[1] hereafter referred to as I. In this paper we apply these techniques to a generalized Duffing oscillator, i.e., an anharmonic oscillator with a general nonlinear force $f(x)$ driven by external noise,

$$\ddot{x} + \alpha\dot{x} + [x + \beta f(x)] = F(t) \tag{1}$$

Here $x(t)$ is the position of the oscillator at time t, α is a damping coefficient, and the natural (linear) frequency of the oscillator is taken to be unity. The

function $f(x)$ is some nonlinear function of x with "strength" β. The driving force or excitation $F(t)$ is stationary noise, which we assume to be a Gaussian delta-correlated force with zero mean, i.e.,

$$\langle F(t) \rangle = 0 \tag{2}$$

$$\langle F(t)F(t') \rangle = 2d\,\delta(t - t') \tag{3}$$

The presence of the nonlinear function $f(x)$ prevents an analytic solution of the problem. To approximate the properties of Eq. (1) by the method of statistical linearization, it is convenient to rewrite it as

$$\ddot{x} + \alpha\dot{x} + \gamma^2 x = F(t) - \Delta(x) \tag{4}$$

where the "error" term $\Delta(x)$ is given by

$$\Delta(x) = \beta f(x) - h_1 x \tag{5}$$

with

$$h_1 \equiv \gamma^2 - 1 \tag{6}$$

The coefficient γ^2, which represents the square of an "equivalent frequency," is determined by minimizing the expectation of the square of the error term with respect to this parameter, i.e., γ^2 is determined from the equation

$$(\partial/\partial h_1)\langle \Delta^2(x) \rangle = 0 \tag{7}$$

where the brackets $\langle\ \rangle$ denote an ensemble average with respect to the equilibrium distribution of the random variable x. Justification for this minimization procedure is given in I. Substitution of Eq. (5) into Eq. (7) yields

$$h_1 = \beta\langle xf(x) \rangle / \langle x^2 \rangle \tag{8}$$

so that

$$\gamma^2 = 1 + [\beta\langle xf(x) \rangle / \langle x^2 \rangle] \tag{9}$$

Statistical linearization consists in dropping the term $\Delta(x)$ from Eq. (4), thus approximating the nonlinear process by a linear one with the equivalent effective frequency defined by Eqs. (6) and (8), i.e.,

$$\ddot{x} + \alpha\dot{x} + \gamma^2 x = F(t) \tag{10}$$

The ensemble averages indicated in Eqs. (8) and (9) are calculated using the equilibrium solution of the linear equation (10). Noting that $F(t)$ is Gaussian δ-correlated noise, it follows that the stationary distribution of x is also Gaussian,

$$P(x) = [1/(2\pi)^{1/2}\sigma_x] \exp(-x^2/2\sigma_x{}^2) \tag{11}$$

with

$$\langle x^2 \rangle = \sigma_x{}^2 = d/\alpha\gamma^2 \tag{12}$$

We restrict our analysis to nonlinear functions $f(x)$ which are odd in x [to assure that the solution $x(t)$ is finite for all t and positive β] and which can be expanded in a power series

$$f(x) = \sum_{j=1}^{\infty} f_j x^{2j+1} \tag{13}$$

Equation (9) then becomes, with the application of Eqs. (11)–(13),

$$d/\alpha\sigma_x{}^2 = 1 + \beta \sum_{j=1}^{\infty} f_j c_j \sigma_x^{2j} \tag{14}$$

where the coefficients c_j are given by

$$c_j = \int_{-\infty}^{\infty} x^{2j+2} \exp(-x^2/2)\, dx \tag{15}$$

Equation (14) must now be solved, in general numerically, for the variance $\sigma_x{}^2$. This solution then yields the "equivalent" frequency γ via Eq. (12). When this equivalent frequency is substituted in Eq. (10) one can then calculate, within the statistical linearization approximation, various statistical properties of the nonlinear oscillator equation (1).

In particular, the autocorrelation function $R_{xx}^{(0)}(t)$ and spectral density $S_{xx}^{(0)}(\omega)$ of the random variable x as calculated from Eq. (10) are

$$R_{xx}^{(0)}(t) \equiv \langle x(\tau + t)x(\tau)\rangle = \frac{d}{\alpha\gamma^2} e^{-(\alpha/2)|t|}\left(\cos \psi|t| + \frac{\alpha}{2\psi} \sin \psi|t|\right) \tag{16}$$

with

$$\psi = (\gamma^2 - \alpha^2/4)^{1/2} \tag{17}$$

and

$$S_{xx}^{(0)}(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega t} R_{xx}^{(0)}(t)\, dt = \frac{2d}{(\gamma^2 - \omega^2)^2 + \omega^2\alpha^2} \tag{18}$$

The superscript zero is used to denote the statistical linearization approximation (10). Equation (18) was also derived by MC[4] by diagrammatic expansion and denoted by them as the "first Kraichnan–Wyld" approximation.

3. CORRECTIONS TO STATISTICAL LINEARIZATION

The method of statistical linearization takes into account the nonlinear error term $\Delta(x)$ of Eq. (4) only in the calculation of the effective frequency. This error term is otherwise ignored. In paper I two procedures are developed which yield higher order corrections to the autocorrelation function obtained

by statistical linearization. Both of these procedures are based on retaining the error term $\Delta(x)$ in Eq. (4) and approximating the contributions of this nonlinear term to the autocorrelation function of $x(t)$. These higher order corrections were first obtained by MC[4] via diagrammatic expansion techniques and termed by them "cumulant discard" and "quasilinear Green's function" approximations.

In the *cumulant discard method*, the Fourier transform of Eq. (1) is multiplied by its complex conjugate to yield the following spectral density equation:

$$|-\omega^2 + i\alpha\omega + 1|^2 S_{xx}(\omega) + \beta^2 S_{ff}(\omega) + \beta[(-\omega^2 + i\alpha\omega + 1)S_{xf}(\omega) + (-\omega^2 + i\alpha\omega + 1)^* S_{fx}(\omega)] = 2d \tag{19}$$

Here $S_{fx}(\omega)$ is the Fourier transform of the autocorrelation function

$$R_{fx}(t) \equiv \langle f[x(\tau + t)]x(\tau)\rangle \tag{20}$$

and the other spectral densities occurring in Eq. (19) are defined analogously in terms of their autocorrelation functions. The unknown higher order correlation functions $R_{fx}(t)$, $R_{xf}(t)$, and $R_{ff}(t)$ arising from Eq. (4) which contribute to Eq. (19) are assumed to be decomposable as joint Gaussian processes. This "quasinormal" assumption is the approximation which leads to the cumulant discard method. In I it is shown that this implies the following relations:

$$R_{xf}(t) = R_{fx}(t) = (a_1/\sigma_x)R_{xx}(t) \tag{21}$$

$$R_{ff}(t) = \sum_{n=1}^{\infty} (a_n{}^2/\sigma_x^{2n})R_{xx}^n(t) \tag{22}$$

Here

$$\sigma_x{}^2 \equiv R_{xx}(0) \tag{23}$$

is the variance, and the coefficients a_n depend on the nonlinearity $f(x)$ and are given by

$$a_n = \int_{-\infty}^{\infty} f(x)P(x)\mathrm{He}_n(x)\,dx \tag{24}$$

The $\mathrm{He}_n(x)$ are Hermite polynomials[6] and $P(x)$ is the stationary Gaussian distribution (11). It should be noted that $a_1 = \sigma_x h_1$, where h_1 is the coefficient of statistical linearization as given in Eq. (8). Fourier transformation of Eqs. (21) and (22) and substitution into Eq. (19) yields upon inverse transformation a nonlinear integral equation for the correlation function $R_{xx}(t)$:

$$R_{xx}(t) = R_{xx}^{(0)}(t) - \beta^2 \sum_{n=2}^{\infty} \frac{a_n{}^2}{\sigma_x^{2n}} \int_{-\infty}^{\infty} dt'\, R_{xx}^{(0)}(t - t')R_{xx}^n(t') \tag{25}$$

The first term on the right-hand side of Eq. (25) is the result of statistical linearization as given by Eq. (16). The other terms are the correction to that result.

The *quasilinear Green's function method* begins by treating Eq. (4) as a linear inhomogeneous equation. Thus the solution $G(t - t')$ of

$$\ddot{G} + \alpha\dot{G} + \gamma^2 G = \delta(t - t') \tag{26}$$

is used as a Green's function to obtain

$$x(\tau) = \int_0^\infty G(t')[F(\tau - t') - \Delta(x(\tau - t'))]\, dt' \tag{27}$$

Multiplying this expression for $x(\tau)$ by a similar one for $x(\tau + t)$ and averaging to form the autocorrelation function $R_{xx}(t)$ yields an expression which contains, on the right-hand side, the higher order correlation functions $R_{\Delta F}$, $R_{F\Delta}$, and $R_{\Delta\Delta}$. Once again these higher order correlation functions are approximated using a quasinormal assumption:

$$R_{\Delta F}(t) = R_{F\Delta}(t) = 0 \tag{28}$$

$$R_{\Delta\Delta}(t) = \frac{(h_1\sigma_x)^2}{\sigma_x^2} R_{xx}(t) - 2h_1\beta\sigma_x \frac{a_1}{\sigma_x^2} R_{xx}(t) + \beta^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\sigma_x^{2n}} R_{xx}^n(t) \tag{29}$$

All the symbols on the right-hand side of Eq. (29) have been defined earlier. The final result of this procedure is then again a nonlinear integral equation for $R_{xx}(t)$:

$$R_{xx}(t) = R_{xx}^{(0)}(t) + \beta^2 \sum_{n=2}^{\infty} \frac{a_n^2}{\sigma_x^{2n}} \int_{-\infty}^{\infty} dt'\, R_{xx}^{(0)}(t - t') R_{xx}^n(t') \tag{30}$$

It should be noted that, as pointed out in I, Eq. (30) differs from Eq. (25) only in the sign of the higher order correction terms.

It is not known *in general* which of these two methods for obtaining the correction terms yields the better approximation for $R_{xx}(t)$. Morton and Corrsin's calculations for the Duffing oscillator indicate that, at least for that particular example, the quasilinear Green's function result, Eq. (30), was in better agreement with their exact computer calculations. In the numerical calculation presented below, we present our results for both cases, i.e., for Eqs. (25) and (30).

In the next section we also carry out explicit calculations and comparisons of the methods discussed here for the more general nonlinear function

$$f(x) = f_1 x^3 + f_2 x^5 + f_3 x^7 \tag{31}$$

For this case the coefficients a_n are explicitly given by

$$a_3^2/\sigma_x^6 = 3!\,(f_1 + 10f_2\sigma_x^2 + 105f_3\sigma_x^4)^2 \tag{32}$$

$$a_5^2/\sigma_x^{10} = 5!\,(f_2 + 21f_3\sigma_x^2)^2 \tag{33}$$

$$a_7^2/\sigma_x^{14} = 7!\,f_3^2 \tag{34}$$

with $a_n = 0$ for $n \geqslant 4$.

4. RESULTS

We present here the results of our calculations of the variances, autocorrelation functions, and spectral densities for the generalized Duffing oscillator (1). To evaluate the accuracy of our calculations, we obviously need to compare them to some exact calculations. Such exact calculations for nonlinear stochastic systems of the type considered here, i.e., Eq. (1), are available only for the spectral density $S_{xx}(\omega)$ of the Duffing oscillator through the analog computer "experiments" of MC.[4] The most accurate analytic results, i.e., the ones most closely in agreement with the analog computer work, were obtained by MC via a very tedious diagrammatic expansion method termed by them the "second Kraichnan–Wyld" approximation. We compare our results with this second Kraichnan–Wyld approximation, to which reference is frequently made in this paper as "exact" results. Bixon and Zwanzig[5] also use the MC calculation as the basis of comparison for their results.

The only quantity which can easily be calculated *exactly* for the nonlinear stochastic equation (1) is the variance σ_x^2. This is a particularly useful quantity with which to compare our approximate results since the method of statistical linearization and corrections thereto are designed to minimize the error in the variance. To calculate the exact variance, we use the Fokker–Planck equation for the probability density $P(x, \dot{x}, t)$,[5]

$$\frac{\partial P}{\partial t} = -\dot{x}\frac{\partial P}{\partial x} + \frac{\partial}{\partial \dot{x}}\{[\alpha\dot{x} + x + \beta f(x)]P\} + d\frac{\partial^2 P}{\partial \dot{x}^2} \tag{35}$$

The equilibrium solution of this equation is

$$P(x, \dot{x}, \infty) = \frac{\exp(-\alpha\dot{x}^2/2d)\exp\{-(\alpha/d)\int_0^x [x' + \beta f(x')]\,dx'\}}{\int_{-\infty}^{\infty} d\dot{x}\int_{-\infty}^{\infty} dx \exp(-\alpha\dot{x}^2/2d)\exp\{-(\alpha/d)\int_0^x [x' + \beta f(x')]\,dx'\}} \tag{36}$$

from which the exact variance,

$$\sigma_x^2 = \int_{-\infty}^{\infty} d\dot{x}\int_{-\infty}^{\infty} dx\, x^2 P(x, \dot{x}, \infty) \tag{37}$$

can readily be obtained.

Table I. Comparisons for σ_x^2

β	σ_x^2(FP)	σ_x^2(SL0)	σ_x^2(SL1)	σ_x^2(SL2)	% Error σ_x^2(SL0)	% Error σ_x^2(SL1)	% Error σ_x^2(SL2)
$f(x) = x^3$, $\alpha = 2$							
1	0.17256	0.16667	0.17140	—	−3.4	−0.67	—
2	0.14480	0.13715	0.14460	0.13158	−5.3	−0.14	−9.1
3	0.12836	0.12013	0.12898	0.11370	−6.4	+0.48	−11.4
5	0.10843	0.10000	0.11075	0.09300	−7.8	+2.1	−14.2
10	0.084034	0.076129	—	—	−9.4	—	—
$f(x) = x^5$, $\alpha = 2$							
0.1	0.23416	0.23141	0.23292	0.23008	−1.2	−0.53	−1.7
0.25	0.22046	0.21350	0.21877	0.20942	−3.1	−0.77	−5.0
0.5	0.20609	0.19456	0.20638	—	−5.5	+0.14	—
1.0	0.18870	0.17272	0.19833	—	−8.5	+5.1	—
3.0	0.15730	0.13623	—	—	−13.4	—	—
$f(x) = (1/3)x^3 + (2/15)x^5 + (17/315)x^7$, $\alpha = 2$							
1	0.19932	0.19159	0.19844	—	−3.9	−0.44	—
2	0.17722	0.16690	0.17842	—	−5.8	+0.68	—
3	0.16279	0.15143	0.16593	—	−7.0	+1.9	—

In Table I we present results for the variance σ_x^2 with damping coefficient $\alpha = 2$ and amplitude $d = 1/2$ for various nonlinearities $f(x)$ and several values of the nonlinearity coefficient β. The exact variance, obtained from Eq. (37), is denoted by σ_x^2(FP). The variance obtained by statistical linearization is denoted by σ_x^2(SL0) and the variances obtained by the quasilinear Green's function and cumulant discard methods are denoted by σ_x^2(SL1) and σ_x^2(SL2) respectively. The percent errors of each method with respect to the exact FP result are also shown. The nonlinearities considered are the Duffing oscillator [$f(x) = x^3$], a fifth-order nonlinearity [$f(x) = x^5$], and the first three terms of the expansion of tan x in a power series. The following conclusions, which we also found to be true for other values of α, can readily be extracted from Table I:

1. Statistical linearization (SL0) gives a variance which is lower in every case than the exact one. The percentage error increases, for all forms of the nonlinearity $f(x)$, with the strength parameter β and depends upon the form of $f(x)$. For small β, where the designation "small" depends on the form of the nonlinearity $f(x)$, the error is of the order of a few percent.

2. The quasilinear Green's function method (SL1) generally reduces the error by a considerable amount, in most cases by an order of magnitude, bringing the variance to within tenths of percent of the exact one.

3. The cumulant discard method invariably gives worse results for the

variance than does statistical linearization for all the cases that we have calculated.

It is important to stress the fact that statistical linearization is a procedure which is trivial to carry out in comparison with the work involved in computing the correction terms. Equations (25) and (30) are nontrivial integral equations for numerical solution and can present difficult computational stability problems for larger nonlinearities. This is evidenced by the dashes in Table I, which represent parameter values and nonlinearities for which we were not able to solve the integral equations via standard iterative techniques. Hence if one is content with accuracy to within a few percent for the variance, our results indicate that statistical linearization is quite sufficient, and the calculation of the correction terms in SL1 will not be worthwhile unless very high accuracy is desired. As we will see below, this conclusion is also valid when we consider autocorrelation functions and spectral densities.

We next present typical results for autocorrelation functions and spectral densities for various nonlinearities. Figures 1 and 2 show respectively the autocorrelation function $R_{xx}(t)$ and its Fourier transform $S_{xx}(\omega)$ for the Duffing oscillator with $f(x) = x^3$. The parameter values $\alpha = 2$ for the damping coefficient and $\beta = 2$ for the nonlinearity strength lead to an equivalent linear oscillator which is overdamped. Hence, $R_{xx}(t)$ exhibits no oscillations and $S_{xx}(\omega)$ peaks at $\omega = 0$. The middle curve on both graphs is the statistical linearization result (SL0); the upper curve corresponds to the quasilinear Green's function method (SL1) and the lower curve to the cumulant discard method (SL2). The main differences in the three methods occur near $t = 0$ for $R_{xx}(t)$ and near $\omega = 0$ for $S_{xx}(\omega)$. From Table I we already know that the upper curve in $R_{xx}(t)$, i.e., the quasilinear Green's

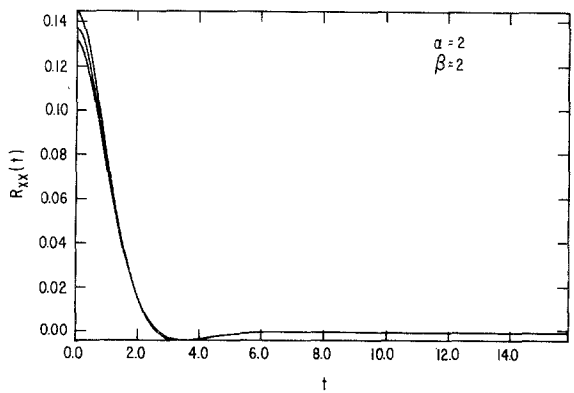

Fig. 1. The autocorrelation function $R_{xx}(t)$ for an overdamped Duffing oscillator $f(x) = x^3$ with $\alpha = 2$ and $\beta = 2$. The central curve (SL0) is calculated from Eq. (16), the upper curve (SL1) from Eq. (30), and the lower curve (SL2) from Eq. (25).

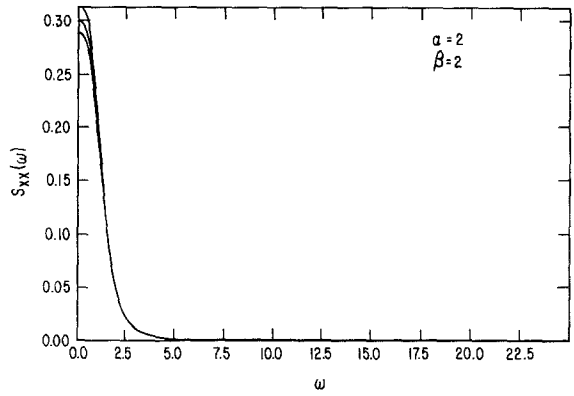

Fig. 2. The spectral density $S_{xx}(\omega)$ corresponding to the autocorrelation function curves $R_{xx}(t)$ of Fig. 1.

function result, gives the best value of $R_{xx}(0) = \sigma_x{}^2$. From comparison with exact results for $S_{xx}(\omega)$ by Morton and Corrsin[4] for slightly different parameter values α and β, we learn that the upper curve (SL1) is the most accurate spectral density curve, while the lower curve (SL2) is the least accurate one. Once again we conclude that the amount of work involved in obtaining the relatively small improvement due to the quasilinear Green's function method is, in most cases, not warranted and that the much simpler method of statistical linearization gives results that are already correct to within a few percent.

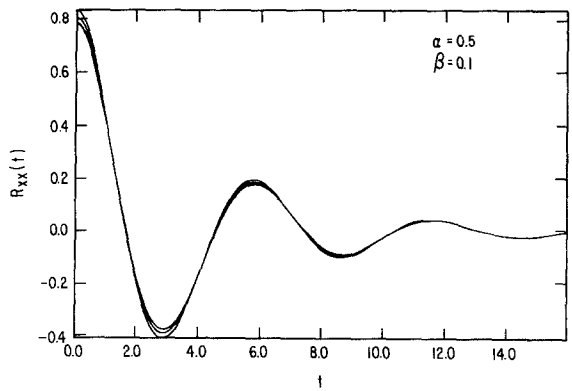

Fig. 3. The autocorrelation function $R_{xx}(t)$ for an underdamped Duffing oscillator $f(x) = x^3$ with $\alpha = 0.5$ and $\beta = 0.1$. The central curve (SL0) is calculated from Eq. (16), the upper curve (SL1) from Eq. (30), and the lower curve (SL2) from Eq. (25).

In Fig. 3 we again present the autocorrelation function $R_{xx}(t)$ for the Duffing oscillator [$f(x) = x^3$], but this time with parameters α and β that lead to an underdamped equivalent linear oscillator. The autocorrelation function therefore oscillates. Again, the middle curve is the result of statistical linearization and the upper and lower curves are results of the quasilinear Green's function (SL1) and cumulant discard (SL2), methods respectively. The differences between the three curves are very small, the largest difference occurring at $t = 0$ and at the first minimum.

It is interesting to note that, in spite of the fact that the SL1 and SL2 curves are solutions of the nonlinear equations (25) and (30), they do not contain higher harmonics of the fundamental frequency of the equivalent linear oscillator as might have been expected. These higher harmonics are, in fact, damped out. To emphasize this point, we show in Fig. 4 the differences $\Delta R_{xx}^{(a)} \equiv R_{xx}^{(\mathrm{SL1})} - R_{xx}^{(\mathrm{SL0})}$ and $\Delta R_{xx}^{(b)} = R_{xx}^{(\mathrm{SL2})} - R_{xx}^{(\mathrm{SL0})}$. The solid curve *a* represents the difference between the autocorrelation functions SL1 and SL0 for parameter values $\alpha = 0.5$ and $\beta = 0.1$, and the solid curve *b* represents the difference between SL2 and SL0 for the same parameter values. We note that these differences ΔR_{xx} are, for all values of t, very small, i.e., $\Delta R_{xx}/R_{xx} = O(10^{-2})$. We further note that the frequency of oscillations of these differences ΔR_{xx} is essentially identical to that of $R_{xx}(t)$ in Fig. 3.

These two observations are consistent with the nonappearance of the higher harmonics in $R_{xx}^{(\mathrm{SL1})}$ and $R_{xx}^{(\mathrm{SL2})}$. The dashed curves *a* and *b* again represent differences ΔR_{xx} for the Duffing oscillator, but now for the overdamped case ($\alpha = 2$, $\beta = 1$). As expected, no oscillations are observed for this overdamped case.

The results presented in Figs. 1–4 and the preceding comments, as well as

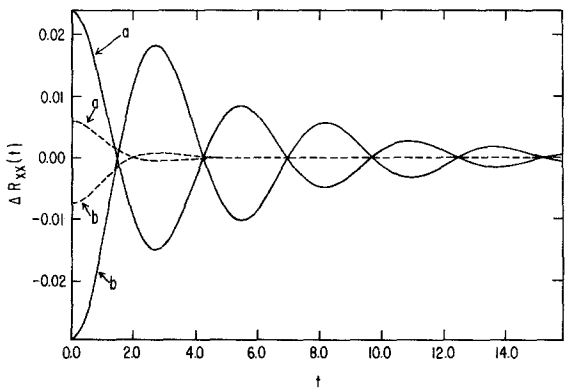

Fig. 4. The differences $\Delta R_{xx}^{(a)} \equiv R_{xx}^{(\mathrm{SL1})} - R_{xx}^{(\mathrm{SL0})}$ (curve *a*) and $\Delta R_{xx}^{(b)} \equiv R_{xx}^{(\mathrm{SL2})} - R_{xx}^{(\mathrm{SL0})}$ (curve *b*) vs. t. The solid lines are for the parameter values $\alpha = 0.5$, $\beta = 0.1$ and the dashed lines for $\alpha = 2$, $\beta = 2$.

our results for other nonlinearities and parameter values not presented here, suggest that there exists a "pseudoscaling" such that

$$R_{xx}^{(\mathrm{SL}1)}(t) \simeq (1 + \lambda) R_{xx}^{(\mathrm{SL}0)}(t) \tag{38}$$

and

$$R_{xx}^{(\mathrm{SL}2)}(t) \simeq (1 - \lambda) R_{xx}^{(\mathrm{SL}0)}(t) \tag{39}$$

with analogous relations for the spectral densities $S_{xx}(\omega)$. In Eqs. (38) and (39) λ is a very slowly varying function of time, almost constant, with $\lambda \ll 1$ for the cases we have considered. Its value can be very roughly estimated from Eq. (25) or Eq. (30) by noting that $|R_{xx}^{n}(t')/\sigma_{2n}^{x}| \ll 1$ for most values of t' and has its largest value (i.e., unity) at $t' = 0$. Using the mean value theorem by taking $R_{xx}^{0}(t)$ out from under the integral yields

$$\lambda \simeq \beta^2 \sum_{n=2}^{\infty} \frac{a_n^2}{\sigma_x^{2n}} \int_{-\infty}^{\infty} dt'\, R_{xx}^{n}(t') \tag{40}$$

This is a simple approximation to the change in $R_{xx}^{(\mathrm{SL}0)}(t)$ brought about by the correction terms in the quasilinear Green's function and cumulant discard methods.

We have also considered the behavior of the autocorrelation function and spectral density for different types of nonlinearities $f(x)$. Figures 5 and 6 show typical results, with $\alpha = 2$ and $\beta = 1$. These curves are all calculated by the quasilinear Green's function method (SL1). Curve c in both figures corresponds to the Duffing oscillator, $f(x) = x^3$. Curve b is for a fifth-order nonlinearity, $f(x) = x^5$. Curve a corresponds to the first three terms in the

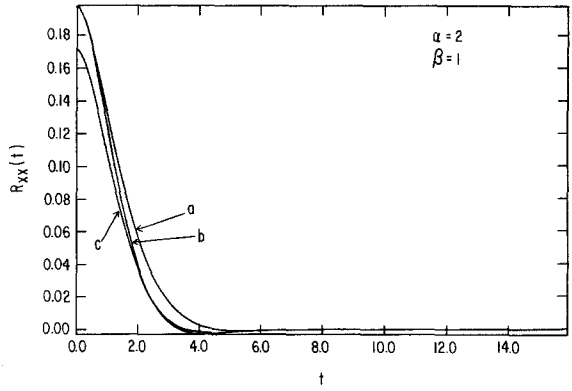

Fig. 5. The autocorrelation function $R_{xx}^{(\mathrm{SL}1)}(t)$ for various nonlinearities $f(x)$ with $\alpha = 2$ and $\beta = 1$. Curve a: $f(x) = x^3/3 + 2x^5/15 + 17x^7/315 \simeq \tan x$; curve b: $f(x) = x^5$; curve c: $f(x) = x^3$.

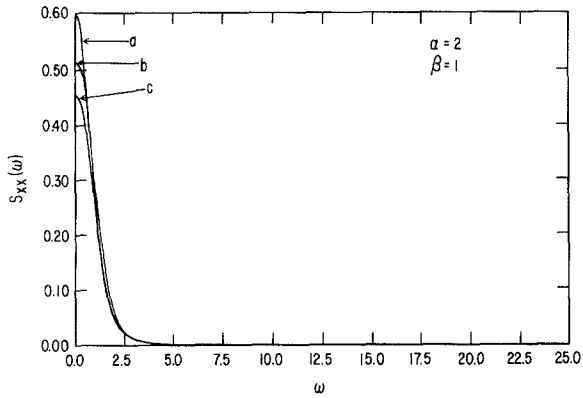

Fig. 6. The spectral densities $S_{xx}(\omega)$ corresponding to the autocorrelation functions $R_{xx}(t)$ of Fig. 5.

power series expansion of $\tan x$, i.e., $f(x) = x^3/3 + 2x^5/15 + 17x^7/315$. It should be noted that the changes in $R_{xx}(t)$ and $S_{xx}(\omega)$ are not very dramatic.

A logical question that may come to mind at this point is the following: How big an error does one make in the calculation of the autocorrelation function and spectral density if one ignores the nonlinearity altogether, i.e., by setting $\beta = 0$? Is this error larger than the difference between statistical linearization and exact results, or between statistical linearization and quasi-linear Green's function results? The answer is that one certainly does much worse when ignoring the nonlinearity altogether. For instance, consider the variances σ_x^2 listed in Table I. The exact variance for a linear oscillator ($\beta = 0$) with damping coefficient $\alpha = 2$ is easily seen from Eq. (37) to be $\sigma_x^2 = 0.25$. Now consider, for example, the case $\beta = 2$ for the Duffing oscillator. From Table I we see that the exact variance, i.e., σ_x^2(FP), is 0.1448, i.e., down from the $\beta = 0$ variance by 42%. Statistical linearization gives a result which differs from the exact result by only 5.3% and the difference between SL0 and SL1 is only 5.2%. The question then remains: Is most of the effect of the nonlinearity contained in the variance? In other words, is the normalized correlation function, $R_{xx}(t)/\sigma_x^2$, fairly accurately calculated if one simply ignores the nonlinearity? The answer is no, as illustrated in Fig. 7. This figure shows $R_{xx}(t)/\sigma_x^2$ with $\alpha = 2$ for the linear oscillator ($\beta = 0$) and the statistical linearization results with $\beta = 0.5$ and $\beta = 2$ for the Duffing oscillator. It is clear from an examination of these curves that also for the normalized autocorrelation function the effect of the nonlinearity is significant and cannot be ignored. Figure 8 shows a similar comparison for the normalized spectral density.

We have found a very simple method which greatly improves the results given by statistical linearization. The implementation of this method is trivial

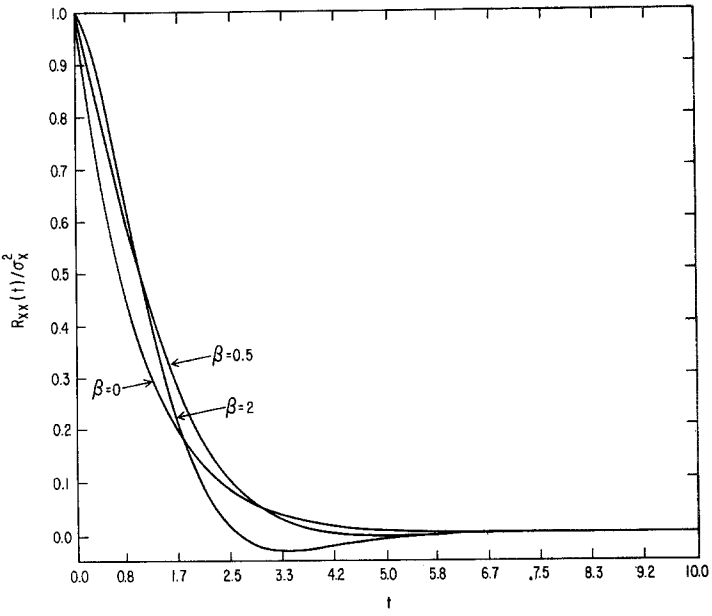

Fig. 7. The normalized autocorrelation function $R_{xx}^{(SL0)}(t)/\sigma_x^2(SL0)$ vs. t for the nonlinearity parameters $\beta = 0$ (harmonic oscillator), $\beta = 0.5$, and $\beta = 2$ and damping coefficient $\alpha = 2$.

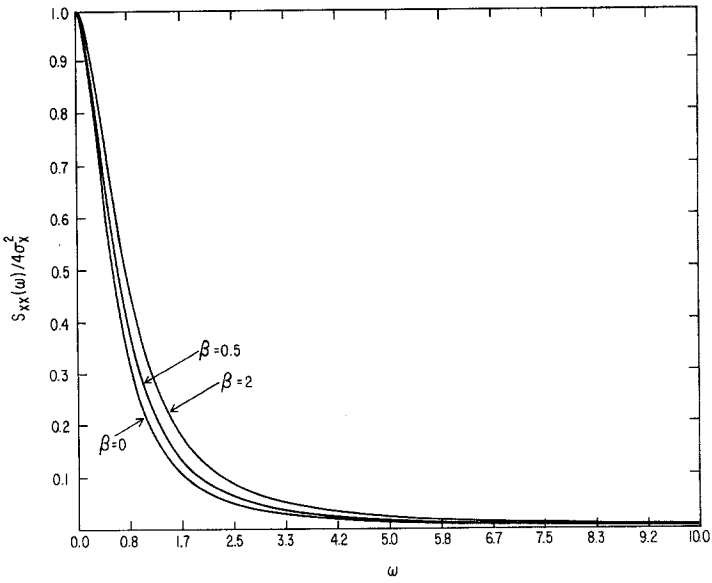

Fig. 8. The normalized spectral density function $S_{xx}(\omega)/4\sigma_x^2$ corresponding to the normalized autocorrelation function curves of Fig. 7.

compared to the work that must be done to obtain the correction terms of the quasilinear Green's function method and, in addition, yields more accurate results than the latter. It consists simply in using the statistical linearization results, Eqs. (16) and (18), but with the variance obtained from the exact Fokker–Planck equation, Eq. (37) in place of the variance obtained by statistical linearization, Eqs. (5)–(8). This procedure yields an effective frequency γ given by

$$\gamma^2 = d/\alpha\sigma_x^2(\text{FP}) \tag{41}$$

in place of Eq. (12). This method therefore automatically gives the exact variance and hence corrects $R_{xx}(t)$ in the region where the approximate results differ most from the exact ones, i.e., near $t = 0$. A more stringent test of this procedure is to analyze the spectral densities obtained by its use. Table II shows the results of such a calculation. These results are for the

Table II. Calculated Values of $S_{xx}(\omega)$

ω	MC KWII "exact"	Statistical linearization (SL0)	Quasilinear Green's function (SL1)	Statistical linearization with σ_x^2(FP)
$f(x) = x^3$, $\alpha = 2$, $\beta = 1$				
0.0	0.48863	0.44444	0.45501	0.47646
0.5	0.40282	0.39024	0.40080	0.41035
1.0	0.22556	0.23529	0.24323	0.23802
1.5	0.10062	0.10458	0.10812	0.10371
2.0	0.04438	0.04494	0.04622	0.04443
2.5	0.02108	0.02102	0.02149	0.02081
3.0	0.01092	0.01084	0.01102	0.01075
3.5	0.00613	0.00608	0.00616	0.00604
4.0	0.00367	0.00365	0.00368	0.00363
4.5	0.00232	0.00231	0.00233	0.00230
5.0	0.00154	0.00153	0.00154	0.00153
$f(x) = x^3$, $\alpha = 0.5$, $\beta = 0.1$				
0.0	0.71639	0.64867	0.66660	0.66841
0.5	1.06344	0.95620	0.98518	0.99057
1.0	3.29662	3.24280	3.36620	3.33560
1.5	0.61345	0.63318	0.66654	0.61846
2.0	0.11607	0.11616	0.12007	0.11480
2.5	0.03765	0.03753	0.03855	0.03727
3.0	0.01614	0.01602	0.01642	0.01594
3.5	0.00813	0.00805	0.00820	0.00802
4.0	0.00453	0.00451	0.00456	0.00450
4.5	0.00273	0.00273	0.00275	0.00272
5.0	0.00175	0.00175	0.00177	0.00175

spectral density for the Duffing oscillator for the two cases: $\alpha = 2$ and $\beta = 1$ (overdamped) and $\alpha = 0.5$ and $\beta = 0.1$ (underdamped). The second column shows the "exact" result obtained by Morton and Corrsin[7] using the second Kraichnan–Wyld approximation (KWII). The third column gives statistical linearization results, the fourth gives the results of the quasilinear Green's function method, and the last column represents the simple modification of statistical linearization discussed above. The most reasonable way to compare these results over the range of ω is to examine the mean square deviation D for the 11 values of ω for which exact values of $S_{xx}(\omega)$ are available:

$$D \equiv \frac{1}{11} \sum_{i=1}^{11} [S_{xx}(\omega_i) - S_{xx}^{\text{exact}}(\omega_i)]^2 \tag{42}$$

We find for the case $\alpha = 2$, $\beta = 1$,

$$\begin{aligned} D(\text{SL0}) &= 2.02 \times 10^{-4} \\ D(\text{SL1}) &= 1.37 \times 10^{-4} \\ D(\text{modified SL0}) &= 3.36 \times 10^{-5} \end{aligned} \tag{43}$$

The modified SL0 is by this criterion 4.1 times better than SL1 and 6.0 times better than SL0. For $\alpha = 0.5$, $\beta = 0.1$ we find

$$\begin{aligned} D(\text{SL0}) &= 1.76 \times 10^{-3} \\ D(\text{SL1}) &= 1.48 \times 10^{-3} \\ D(\text{modified SL0}) &= 8.33 \times 10^{-4} \end{aligned} \tag{44}$$

Here the modified SL0 is an improvement by a factor of 1.8 over SL1 and by a factor of 2.1 over SL0. In both cases, the modified statistical linearization gave better agreement with the exact results, as judged by the mean square deviation, than the quasilinear Green's function method (SL1).

5. CONCLUSIONS

1. Statistical linearization is a simple and effective method to obtain good approximate results for autocorrelation functions and spectral densities for nonlinear systems described by Eq. (1). This conclusion holds for relatively large ranges of damping (α) and nonlinearity (β) coefficients and for a variety of nonlinearities $f(x)$.

2. The cumulant discard method, Eq. (30), invariably yielded results which were worse than statistical linearization.

3. The quasilinear Green's function method, Eq. (25), consistently gave results which were more accurate than those of statistical linearization. The computational effort involved in solving Eq. (25) is, however, very large and probably not worthwhile for the small gain in accuracy obtained.

4. The method which seems to give the best results, i.e., the best approximations to the known exact results, is the modified statistical linearization method in which the formal results of statistical linearization are used, but with an effective frequency determined by the exact variance as calculated from the Fokker–Planck equation. This method has the very important advantage of great computational simplicity as compared to all other attempts to improve the results of statistical linearization.

NOTE ADDED IN PROOF

We thank Prof. Stephen H. Crandall for pointing out to us that statistical linearization with the exact variance (the procedure we call modified SL0) has been used previously for several systems by Caughey and Dienes[8] and also by Crandall and co-workers.[9] These authors also found this modification to be a significant improvement over simple statistical linearization.

ACKNOWLEDGMENT

We wish to thank Mr. Raph Janda of Physical Dynamics, Inc., La Jolla, California, who carried out very ably most of the computer calculations presented in this paper.

REFERENCES

1. A. B. Budgor, *J. Stat. Phys.*, this issue, preceding paper.
2. T. K. Caughey, *J. Acoust. Soc. Am.* **35**:1706 (1963); *Adv. Appl. Mech.* **11**:209 (1971).
3. S. H. Crandall, Nonlinear Vibration Problems, *Zagadnienia Drgań Nielinowych* **14**:39 (1973); Nonlinear Problems in Random Vibrations, presented at the 7th International Conference on Nonlinear Oscillations, Berlin, Sept. 1975.
4. J. B. Morton and S. Corrsin, *J. Stat. Phys.* **2**:153 (1970).
5. M. Bixon and R. Zwanzig, *J. Stat. Phys.* **3**:245 (1971).
6. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards Applied Math. Series, No. 55), U.S. Government Printing Office, Washington, D.C. (1968), p. 778.
7. J. B. Morton, private communications.
8. T. K. Caughey and J. K. Dienes, *J. Appl. Phys.* **32**:2476 (1961).
9. S. H. Crandall, S. S. Lee, and J. H. Williams, Jr., *J. Appl. Mech.* **41**:1094 (1974); S. H. Crandall and S. S. Lee, to appear in *Ingenieur-Archiv.*